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# Newton–Cartan (super)gravity as a non-relativistic limit

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## Abstract

We define a procedure that, starting from a relativistic theory of supergravity, leads to a consistent, non-relativistic version thereof. As a first application we use this limiting procedure to show how the Newton–Cartan formulation of non-relativistic gravity can be obtained from general relativity. Then we apply it in a supersymmetric case and derive a novel, non-relativistic, off-shell formulation of three-dimensional Newton–Cartan supergravity.

Keywords: Newton–Cartan gravity, non-relativistic limit, non-relativistic supergravity

## 1. Introduction

Recent advances in the study of non-relativistic field theories have spurred a renewed interest in Newton–Cartan geometry. Originally devised as a topic in differential geometry that treats Newtonian spacetimes with a notion of absolute time and space, Newton–Cartan geometry has mostly been discussed in the context of Newton–Cartan gravity [1, 2]. The latter refers to a generally covariant and geometric reformulation of Newtonian gravity that mimics general relativity as much as possible (see chapter 12 of [3] for a textbook exposition). Newton–



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Cartan geometry then plays a similar role for Newton–Cartan gravity as Riemannian geometry does for general relativity<sup>3</sup>.

Recently, Newton–Cartan geometry has also been considered in entirely different contexts. In particular, it has been crucial in work by Son and collaborators on the fractional quantum Hall effect [5–7]. Here, Newton–Cartan geometry and diffeomorphism invariance are used as a guiding principle to construct an effective action for an external gauge-field and a background metric source, that can be used to find the electromagnetic and gravitational response of a quantum Hall fluid. It was argued that this effective action can capture universal features of the quantum Hall effect, other than the quantized Hall conductivity that is determined by its lowest order Chern–Simons term. For instance, the effective action naturally includes a so-called Wen–Zee term [8], that describes a coupling between the gauge-field and spatial curvature and that universally encodes the Hall viscosity.

Newton–Cartan geometry has also been instrumental in the context of Lifshitz holography, that attempts to define a gravitational dual for non-relativistic field theories that are invariant under time and spatial translations, spatial rotations and anisotropic dilatations. The putative gravitational dual is formulated around a so-called ‘asymptotically locally Lifshitz’ spacetime and the boundary geometry of such a spacetime is described by Newton–Cartan geometry with torsion [9]. This observation regarding the boundary geometry of Lifshitz spacetimes was subsequently used in [10–13] to e.g. define the boundary stress tensor and to calculate holographic Ward identities in Lifshitz holography.

The above developments have motivated recent studies on how non-relativistic field theories can be appropriately coupled to arbitrary Newton–Cartan backgrounds (with or without torsion) [14–19]. In most of the studies that appeared in the literature, Newton–Cartan geometry is considered in a metric formalism, that features two degenerate metrics of rank 1 and rank  $d$  (with  $d$  the number of spatial dimensions), used to measure temporal and spatial distances respectively. In such a formalism, parallel transport is defined via an affine connection, that can be defined via metric compatibility. While the metric formulation provides a clear definition of Newton–Cartan geometry, for many practical applications an equivalent vielbein formulation is often more suitable. Such a vielbein formulation introduces local spatial rotations and Galilean boosts as well as associated spin-connections and can thus be very useful e.g. when considering couplings to fermions or incorporating supersymmetry.

At first sight, studying supersymmetric non-relativistic field theories on arbitrary Newton–Cartan backgrounds might seem rather academic. In the relativistic case however, there exist powerful localization techniques that allow one to extract exact results for supersymmetric field theories on curved backgrounds [20] (see also the lecture notes [21] and references therein). A convenient procedure for determining on which curved backgrounds supersymmetric theories can be formulated and what supersymmetry algebra such theories exhibit, was given in [22]. This procedure essentially consists of coupling the flat space theory to off-shell supergravity, choosing a classical background (specified by the metric and arbitrary values for the auxiliary fields in the off-shell multiplet) and sending the Planck mass to infinity. If one wishes to investigate whether these localization techniques can be extended to non-relativistic field theories, one thus needs to obtain off-shell realizations of non-relativistic supergravity. For this, an appropriate vielbein formulation of Newton–Cartan geometry is essential.

<sup>3</sup> Non-relativistic gravitational theories are not unique, for example the gravitational background that is used to describe the Newton–Cartan point-particle is different from the background that describes non-relativistic branes, see e.g. [4]. In this work we only consider backgrounds of the first kind, i.e. ‘particle’ backgrounds, which feature more prominently in the literature.

Torsionless Newton–Cartan geometry, as it appears in Newton–Cartan gravity, was developed in terms of vielbeine in [23, 24]. It was in particular shown that this geometry can be obtained via a gauging of the Bargmann algebra, i.e. the central extension of the Galilei algebra. In this gauging, one introduces gauge-fields for every generator of the Bargmann algebra, along with constraints on some of the gauge covariant curvatures. The latter are interpreted as torsion conditions or as constraints that allow one to express local time and spatial translations as diffeomorphisms. The temporal and spatial vielbeine then appear as the gauge-fields associated to time and spatial translations and their transformation rules are determined by the Bargmann algebra. Crucially, the vielbein formulation obtained in this way also includes an independent one-form that is interpreted as a gauge-field for central charge transformations. The role of this extra central charge gauge-field in constructing couplings to arbitrary curved non-relativistic backgrounds has been discussed recently in [18, 19]. The gauging procedure has been extended to obtain an on-shell supergravity version of three-dimensional Newton–Cartan gravity in [24]. In [25], it was moreover shown that the torsional Newton–Cartan geometry that appears in Lifshitz holography can be obtained from gauging the Schrödinger algebra, a conformal extension of the Bargmann algebra. Also in that case, the inclusion of a central charge gauge-field is necessary.

While the gauging procedure provides an effective tool to construct a vielbein formulation of bosonic Newton–Cartan geometry and gravity, it is not always sufficient to obtain supersymmetric generalizations thereof. In those cases, the vielbeine and central charge gauge-fields are part of a supermultiplet, that might contain extra fields, that cannot be interpreted as gauge-fields of an underlying non-relativistic superalgebra. This is in particular, but not exclusively, the case when off-shell formulations are considered. When considering supersymmetry, a different procedure to obtain the field content and transformation rules of a supermultiplet that contains the vielbeine and the central charge gauge-field is therefore necessary. The aim of this paper is to provide such a procedure and to illustrate it in various examples.

The procedure developed here essentially consists of taking a non-relativistic limit of vielbein formulations of relativistic (super)gravity. We will show how such a limit can be defined and implemented in a consistent manner. This limit in particular sheds light on how Newton–Cartan gravity in the vielbein formulation can be obtained as a non-relativistic limit of general relativity. Apart from elucidating how Newton–Cartan geometry descends from relativistic Riemannian geometry (see [19, 26–29] for early and more recent work on this), the limiting procedure also has the advantage that it can be used to obtain versions of Newton–Cartan geometry and gravity that cannot be obtained via gauging. We will in particular use it to obtain an off-shell version of the three-dimensional Newton–Cartan supergravity theory constructed in [24]. We should stress that the limiting procedure we discuss in this work can be used to obtain versions of Newton–Cartan geometry, that descend from relativistic geometries but that it is *a priori* not clear that every non-relativistic geometry can be obtained in this way. The torsional Newton–Cartan geometry constructed in [25] for instance is based on the Schrödinger algebra, that cannot be obtained via a contraction of a relativistic conformal algebra. This geometry might thus furnish an example of a non-relativistic geometry that does not descend from a relativistic one in an easy manner.

This paper is organized as follows. In section 2 we explain the general procedure that allows us to obtain non-relativistic geometries from relativistic ones. We illustrate this method with several examples that have been constructed in the literature using other methods. As a first example, we will show how the vielbein formulation of torsionless Newton–Cartan geometry of [23] can be obtained in this way. Our second example will deal with the on-shell three-dimensional Newton–Cartan supergravity theory of [24]. As a third example, we will

show how non-relativistic particle actions can be obtained from relativistic ones, using the limiting procedure. In section 3, we will use the same procedure to obtain a supersymmetric generalization of Newton–Cartan gravity that has not yet appeared in the literature, namely an off-shell formulation of three-dimensional Newton–Cartan supergravity. Finally, we conclude in section 4 and give an outlook on various problems that could be handled using the methods described in this paper.

## 2. The road to non-relativistic supergravity

In this section we discuss how to derive non-relativistic geometries and gravity theories from relativistic ones. We first describe our procedure, which amounts to taking a non-relativistic limit in a consistent manner, in section 2.1. We then illustrate this limiting procedure in three examples. In section 2.2 we re-derive the results of [23] regarding the vielbein description of torsionless Newton–Cartan gravity in arbitrary dimensions and in section 2.3 we re-derive the on-shell three-dimensional Newton–Cartan supergravity theory of [24]. In section 2.4, we apply the limiting procedure to derive the results obtained in [30] for the non-relativistic superparticle in a curved background.

### 2.1. The general procedure

The method used in this paper can be viewed as an extension of the contraction of a relativistic spacetime symmetry algebra to a non-relativistic one. In particular, its aim is to mimic the algebra contraction to obtain an irreducible multiplet of fields that represents the contracted non-relativistic algebra starting from an irreducible multiplet of the parent relativistic algebra.

Recall that when performing a standard Inönü–Wigner contraction of a symmetry algebra one first redefines the generators of the algebra, by taking linear combinations of the original generators with coefficients that involve a contraction parameter  $\omega$ . The contracted algebra is then obtained by calculating commutators of the redefined generators, re-expressing the result in terms of them and taking  $\omega \rightarrow \infty$  in the end. An Inönü–Wigner contraction performed in this way does not change the number of generators. Moreover, when considering finite  $\omega$  the algebra of redefined generators is equivalent to the original one.

When extending this algebra contraction to an irreducible multiplet of fields that represents the parent relativistic algebra, it is useful to divide the fields in three categories. A first category consists of independent fields that can be viewed as gauge-fields that are associated to certain generators of the algebra. For instance, the vielbein of general relativity roughly plays the role of the gauge-field of local translations [31]. A second category comprises gauge-fields that are not independent, but that instead depend on other fields in the multiplet. This is the way in which the spin-connection of general relativity can be viewed, namely as a dependent gauge-field for local Lorentz transformations. Finally, the last category contains independent fields that cannot be interpreted as gauge-fields of the underlying spacetime symmetry algebra. This is for instance the case when considering off-shell supergravity multiplets, where typically auxiliary fields are necessary to guarantee the closure of the commutator algebra and, in the relativistic case, to ensure that the number of bosonic and fermionic degrees of freedom match. For simplicity, we will call all such additional fields, that are not gauge-fields corresponding to a generator, auxiliary fields. The fields that can be viewed as gauge-fields corresponding to the generators of the algebra are in general subject to constraints. Some of these constraints are called ‘conventional’ and merely serve to express the dependent gauge-fields in terms of the independent ones. These constraints are identically

satisfied, once the explicit expressions for the dependent fields are plugged in. There will in general also be a second type of ‘un-conventional’ constraints, that are not identically satisfied. This second type of constraints will play a crucial role in ensuring consistency of the limiting procedure.

The first step of the limiting procedure consists of extending the redefinition of the algebra generators, with the contraction parameter  $\omega$ , to all fields and symmetry parameters of the relativistic multiplet. Let us therefore denote the parent algebra generators collectively by  $T_A$ , the parent symmetry parameters by  $\xi^A$  and the parent fields that correspond to gauge-fields by  $A_\mu^A$ . The redefinition of the generators  $T_A$  to generators  $\tilde{T}_A$ , that involves  $\omega$  and defines the contraction, can then be extended to redefinitions of  $\xi^A$  to  $\tilde{\xi}^A$  and of  $A_\mu^A$  to  $\tilde{A}_\mu^A$  such that<sup>4</sup>

$$\xi^A T_A = \tilde{\xi}^A \tilde{T}_A, \quad A_\mu^A T_A = \tilde{A}_\mu^A \tilde{T}_A. \quad (1)$$

This defines the tilded parameters and fields in terms of the original ones and the contraction parameter  $\omega$ . For finite  $\omega$ , this redefinition merely corresponds to a field redefinition and the redefined multiplet is equivalent to the original one. Note that (1) involves both independent and dependent gauge-fields. For the dependent fields, one should take special care that the redefinition suggested by (1) is consistent with the one obtained by performing the redefinitions in the explicit expressions of the dependent fields in terms of the independent ones and that their  $\omega \rightarrow \infty$  limit is well-defined. This amounts to a non-trivial consistency check, that we will discuss further in the next paragraph. Equation (1) does not determine the redefinitions of the auxiliary fields. These are found by examining all transformation rules in terms of redefined gauge-parameters and gauge-fields and by requiring that no term in the transformation rules diverges when taking the limit  $\omega \rightarrow \infty$ . As we will see in specific examples, this can typically be achieved by suitably rescaling the auxiliary fields with the contraction parameter  $\omega$ .

Once all redefinitions have been determined, one can apply them in the transformation rules of all independent fields and in the un-conventional constraints. Their non-relativistic versions are then obtained by sending  $\omega \rightarrow \infty$ . For properly chosen redefinitions, no divergences are encountered here. One does however need to check whether other quantities are finite in this limit. In particular, one needs to examine the expressions of the dependent fields in terms of the independent ones and check whether one obtains a finite result, consistent with the redefinition implied by (1), in the limit  $\omega \rightarrow \infty$ . Typically, terms that are proportional to positive powers of  $\omega$ , and hence blow up in the  $\omega \rightarrow \infty$  limit, do show up in the expressions for the dependent gauge-fields. One can however use the un-conventional constraints, written in terms of redefined fields, to replace these by terms that are finite or vanishing in the  $\omega \rightarrow \infty$  limit. In this way, the relativistic dependent gauge-fields have a well-defined  $\omega \rightarrow \infty$  limit, consistent with (1) and lead to the correct finite expressions for the non-relativistic dependent gauge fields.

Once all non-relativistic transformation rules, dependent gauge-fields and constraints have been found in this way, one needs to check whether the constraints form a consistent set. This involves varying all non-trivial constraints found so far under all symmetry transformations and checking that they form a closed set.

Finally, we mention that the limiting procedure can lead to the elimination of a number of auxiliary fields. This is due to the fact that we are interested in obtaining an irreducible multiplet. Loosely speaking, the non-relativistic theory can have less equations of motion than the relativistic one. The number of auxiliary fields that are needed to realize the non-

<sup>4</sup> Strictly speaking, the second equation only holds up to terms that are subleading in  $\omega$ . This will be clarified in the specific examples, see e.g. the discussion leading to equation (26).

relativistic algebra can therefore differ from the number that is needed to realize the relativistic algebra. This explains why some of the auxiliary fields can be eliminated in the limiting process.

We can summarize the procedure in the following way:

- I. We first write the relativistic gauge-fields in terms of new redefined ones, using a contraction parameter  $\omega$ . This field redefinition is dictated by the same redefinition that one performs on the generators to define the contraction of the algebra. The new fields will become the proper non-relativistic fields after we have taken the  $\omega \rightarrow \infty$  limit. At this point the scaling of the auxiliary fields is still arbitrary.
- II. Using the above redefinitions and taking the limit  $\omega \rightarrow \infty$  we can derive a first set of non-relativistic constraints by taking the limit  $\omega \rightarrow \infty$  of the relativistic unconventional ones.
- III. In a third step we derive the transformation rules of all fields. Requiring that no terms diverge in the limit  $\omega \rightarrow \infty$  fixes the scalings of the auxiliary fields. At this point, we can check the limit of the dependent gauge-fields, such as, e.g., the spin-connection field. Requiring that they have a well-defined limit may involve the use of the un-conventional constraints, written in terms of the redefined fields, in order to replace divergent terms by terms with a proper limit.
- IV. In this step we check whether the constraints found in step II form a closed set under the different symmetry transformations or whether we are forced to introduce additional constraints. An example where many new constraints are found by continuous variation under supersymmetry is given by the chain of constraints in equations (61)–(63).
- V. The number of auxiliary fields that are needed in the non-relativistic case may be less than the number that is needed in the relativistic case. In such cases, in order to obtain an irreducible multiplet, we eliminate the redundant auxiliary fields. This occurs, for instance, in the example of section 3.

In the next subsections, we will illustrate the above limiting procedure by applying it to re-derive various results on Newton–Cartan geometry and (super)gravity that have been obtained in the literature using other methods.

## 2.2. Example 1: Newton–Cartan geometry and gravity from general relativity

In this section we illustrate the limiting procedure that we just described, to obtain the vielbein description of Newton–Cartan gravity of [23] from the vielbein formulation of general relativity. We will pay specific attention to how the transformation rules of the Newton–Cartan fields arise, to how the limiting procedure leads to the correct constraints that these fields have to obey and to how the correct dependent gauge-fields are obtained. First, we summarize in section 2.2.1 our starting point, namely the kinematics of general relativity as a gauging of the Poincaré algebra. Then, using those formulas, we deduce Newton–Cartan geometry and gravity in section 2.2.2.

**2.2.1. The kinematics of general relativity.** It is well-known that the vielbein formulation of general relativity can be viewed as a gauging of the Poincaré algebra [31]. Here we will briefly summarize this gauging procedure. Note that this leads to the kinematics of general relativity. In order to obtain the dynamics, one has to supplement the formulas collected here with the Einstein equations. If one is however only interested in geometrical aspects and not so much in dynamics, one need not do so. This section will serve as the starting point of our



limiting procedure, that will lead to the formulation of Newton–Cartan gravity as obtained via a gauging of the Bargmann algebra in [23].

The kinematics of general relativity in a  $(d + 1)$ -dimensional spacetime is described by the vielbein  $E_\mu^A$  and spin-connection  $\Omega_\mu^{AB}$ ,  $A, B = 0, \dots, d$ , that can be viewed as gauge-fields associated to translations  $\hat{P}_A$  and Lorentz rotations  $M_{AB}$ . Under diffeomorphisms (with parameter  $\xi^\mu$ ) and Lorentz rotations (with parameter  $\lambda^{AB}$ ) these fields transform as

$$\delta E_\mu^A = \xi^\rho \partial_\rho E_\mu^A + E_\rho^A \partial_\mu \xi^\rho + \lambda^A_B E_\mu^B, \quad (2)$$

$$\delta \Omega_\mu^{AB} = \xi^\rho \partial_\rho \Omega_\mu^{AB} + \Omega_\rho^{AB} \partial_\mu \xi^\rho + \partial_\mu \lambda^{AB} + 2 \lambda^{[A}_C \Omega_\mu^{CB]}. \quad (3)$$

One may define gauge covariant curvatures, dictated by the structure constants of the Poincaré algebra, as follows:

$$R_{\mu\nu}^A(E) = 2 \partial_{[\mu} E_{\nu]}^A - 2 \Omega_{[\mu}^A{}^B E_{\nu]}^B, \quad (4)$$

$$R_{\mu\nu}^{AB}(\Omega) = 2 \partial_{[\mu} \Omega_{\nu]}^{AB} - 2 \Omega_{[\mu}^A{}^C \Omega_{\nu]}^{CB}. \quad (5)$$

The spin-connection  $\Omega_\mu^{AB}$  is not an independent field; rather it is given in terms of  $E_\mu^A$  by solving the torsion constraint

$$R_{\mu\nu}^A(E) = 0. \quad (6)$$

This constraint also allows one to replace local translations by general coordinate transformations. The solution of (6) is given by

$$\Omega_\mu^{AB}(E) = -2 E^{\rho[A} \partial_{[\mu} E_{\rho]}^{B]} + E_{\mu C} E^{\rho A} E^{\nu B} \partial_{[\rho} E_{\nu]}^C. \quad (7)$$

Note that imposing the torsion constraint (6) also implies that the Riemann curvature tensor (5) identically satisfies the Bianchi identity

$$R_{[\mu\nu\rho]}^B(\Omega(E)) = R_{[\mu\nu}^{AB}(\Omega(E)) E_{\rho]A} = 0. \quad (8)$$

The above defines the kinematics of general relativity. The dynamics can be obtained by imposing equations of motion, i.e. putting the theory on-shell. In general relativity this is done by imposing the Einstein equations

$$E^\mu{}_A R_{\mu\nu}^{AB}(\Omega(E)) = 0. \quad (9)$$

The limiting procedure by which we will obtain non-relativistic geometry and gravity is an extension of contractions of relativistic symmetry algebras. As explained in [23], torsionless Newton–Cartan geometry and gravity is linked to the Bargmann algebra. In order to obtain the Bargmann algebra from an Inönü–Wigner contraction, one should start from a direct sum of the Poincaré algebra with an abelian factor with generator  $\mathcal{Z}$ . In this way, one ensures that the algebra before contraction has the same number of generators as the Bargmann algebra. The abelian factor is represented by an abelian gauge-field  $M_\mu$ , that transforms under diffeomorphisms and abelian gauge transformations (with parameter  $\Lambda$ ) as follows:

$$\delta M_\mu = \xi^\rho \partial_\rho M_\mu + M_\rho \partial_\mu \xi^\rho + \partial_\mu \Lambda. \quad (10)$$

The curvature of  $M_\mu$  is given by

$$F_{\mu\nu}(M) = 2 \partial_{[\mu} M_{\nu]}. \quad (11)$$

In order to take the non-relativistic limit we will need to impose constraints on this curvature. For example, if we consider the dynamics of general relativity, we do not want to add extra degrees of freedom, apart from the ones contained in the vielbein. In this case, we will thus set



the curvature  $F_{\mu\nu}(M)$  to zero so that  $M_\mu$  corresponds to a pure gauge-field. Another example where we will constrain  $F_{\mu\nu}(M)$  to be zero, will appear when we discuss on-shell supersymmetry. There, this constraint will ensure that the equality of bosonic and fermionic degrees of freedom in the relativistic multiplet is not upset, after  $M_\mu$  is added by hand to an existing on-shell multiplet. Even if we are only interested in the kinematics we cannot allow for a completely arbitrary  $F_{\mu\nu}(M)$ . As we will see in the following we have to constrain the spatial projection of  $F_{\mu\nu}(M)$  to be zero in order to take the non-relativistic limit in a consistent manner. In particular, this will be crucial to obtain finite expressions for the non-relativistic dependent spin-connections as limits of the relativistic one.

In the following section, we will apply the limiting procedure to the formulas collected above.

**2.2.2. Newton–Cartan gravity from relativistic gravity.** In this first example, we will extend the contraction that gives the Bargmann algebra from the Poincaré algebra, to the vielbein and spin-connection of general relativity, along the lines of section 2.1. As explained above, the contraction and its extension involve redefining generators and fields using a contraction parameter  $\omega$ . We will be careful in distinguishing quantities that are merely redefined relativistic ones, for which  $\omega$  is finite, from non-relativistic ones, that are obtained in the limit  $\omega \rightarrow \infty$ . In particular, we will denote the former ones with a tilde, whereas for the latter the tilde will be dropped.

Let us first briefly recall the Inönü–Wigner contraction of the Poincaré algebra to the Bargmann algebra. The contraction is best described by starting from a direct sum of the Poincaré algebra (with translation generators  $\hat{P}_A$  and Lorentz generators  $M_{AB}$ ) with an abelian factor (with generator  $\mathcal{Z}$ )<sup>5</sup>. Starting from the Poincaré algebra

$$[\hat{P}_A, M_{BC}] = 2 \eta_{A[B} \hat{P}_{C]}, \quad [M_{AB}, M_{CD}] = 4 \eta_{[A[C} M_{D]B]}, \quad (12)$$

supplemented with the generator  $\mathcal{Z}$ , we redefine the generators using a contraction parameter  $\omega$  as follows<sup>6</sup>

$$\hat{P}_0 \rightarrow \frac{1}{2\omega} \tilde{H} + \omega \tilde{\mathcal{Z}}, \quad \mathcal{Z} \rightarrow \frac{1}{2\omega} \tilde{H} - \omega \tilde{\mathcal{Z}}, \quad M_{a0} \rightarrow \omega \tilde{G}_a, \quad (13)$$

where we have split the spacetime index  $A$  into a time-like 0-index and spatial  $a, b$ -indices. Note that the spatial translations  $\hat{P}_a$  and spatial rotations  $M_{ab}$  are left untouched, i.e.  $\hat{P}_a = \tilde{P}_a$  and  $M_{ab} = \tilde{M}_{ab}$ . We will in the following denote  $\tilde{M}_{ab} = \tilde{J}_{ab}$  to conform to earlier literature. Calculating commutators of  $\tilde{H}$ ,  $\tilde{P}_a$ ,  $\tilde{G}_a$ ,  $\tilde{J}_{ab}$ ,  $\tilde{\mathcal{Z}}$ , re-expressing the result in terms of the same generators and taking  $\omega \rightarrow \infty$ , we obtain the Bargmann algebra with the non-vanishing commutators

$$\begin{aligned} [P_a, J_{bc}] &= 2 \delta_{a[b} P_{c]}, & [J_{ab}, J_{cd}] &= 4 \delta_{[a[c} J_{d]b]}, \\ [G_a, J_{bc}] &= 2 \delta_{a[b} G_{c]}, & [H, G_a] &= P_a, & [P_a, G_b] &= \delta_{ab} \mathcal{Z}. \end{aligned} \quad (14)$$

In order to derive Newton–Cartan gravity, as obtained via the gauging of (14) in [23], we extend the contraction (13) to the vielbein  $E_\mu^A$ , spin-connection  $\Omega_\mu^{AB}(E)$  and abelian gauge-field  $M_\mu$  of the previous section. In particular, we make the following redefinition for the

<sup>5</sup> We have indicated the relativistic translation generators with a hat to distinguish them from the redefined generators (indicated with a tilde) and the Bargmann generators (with no hat or tilde).

<sup>6</sup> Note that this redefinition corresponds to a non-relativistic particle limit where time is singled out as a special direction. One can define more general  $p$ -brane limits where one time and  $p$  spatial directions are singled out, see e.g. [4, 32–34] and the comment in footnote 3.

relativistic vielbein:

$$E_\mu^A = \delta_0^A \left( \omega \tilde{\tau}_\mu + \frac{1}{2\omega} \tilde{m}_\mu \right) + \delta_a^A \tilde{e}_\mu^a, \quad (15)$$

where  $\tilde{\tau}_\mu$ ,  $\tilde{e}_\mu^a$  and  $\tilde{m}_\mu$  will be identified, in the limit  $\omega \rightarrow \infty$  (where we will drop the tilde), as the independent gauge-fields of Newton–Cartan geometry and gravity. It is convenient to define fields  $\tilde{e}^\mu_a$ ,  $\tilde{\tau}^\mu$  as follows:

$$\tilde{e}^\mu_a \tilde{e}_\mu^b = \delta_a^b, \quad \tilde{\tau}^\mu \tilde{\tau}_\mu = 1, \quad \tilde{\tau}^\mu \tilde{e}_\mu^a = \tilde{\tau}_\mu \tilde{e}^\mu_a = 0, \quad \tilde{e}^\rho_a \tilde{e}_\mu^a = \delta_\mu^\rho - \tilde{\tau}_\mu \tilde{\tau}^\rho. \quad (16)$$

In the limit  $\omega \rightarrow \infty$ , the fields  $\tau_\mu$ ,  $\tau^\mu$ ,  $e_\mu^a$  and  $e^\mu_a$  can be used to define two separate non-degenerate Galilei-invariant metrics, one in the time direction and one in the three spatial directions. These will not be needed in the present discussion. Using the defining relations (16) the following expansion of the relativistic inverse vielbein is obtained

$$E^\mu_A = \delta_A^a \left[ \tilde{e}^\mu_a + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right] + \frac{1}{\omega} \delta_A^0 \left[ \tilde{\tau}^\mu + \mathcal{O}\left(\frac{1}{\omega^2}\right) \right]. \quad (17)$$

Note that in this expression, we have only explicitly given the terms of leading order in  $\omega$ . There are in principle an infinite number of corrections of lower order in  $\omega$ , that we have denoted by  $\mathcal{O}(1/\omega^2)$  and that will not be needed in the following (as they will not contribute in the  $\omega \rightarrow \infty$  limit).

The abelian gauge-field  $M_\mu$  is redefined as follows

$$M_\mu = \omega \tilde{\tau}_\mu - \frac{1}{2\omega} \tilde{m}_\mu. \quad (18)$$

The relativistic spin-connection  $\Omega_\mu^{AB}(E)$  is a dependent field, determined by the torsion constraint (6). As already mentioned under equation (11), we now have to impose a constraint on the curvature of the relativistic gauge-field  $M_\mu$  in order to lower the powers of  $\omega$  in certain terms in the expression for  $\Omega_\mu^{AB}(E)$ , see equation (25) below, such that the limit can be taken. In particular we mentioned the following choices

$$\text{dynamical : } F_{\mu\nu}(M) = 0, \quad (19)$$

$$\text{kinematical : } \tilde{e}^\mu_a \tilde{e}^\nu_b F_{\mu\nu}(M) = \tilde{F}_{ab}(M) = 0. \quad (20)$$

The last choice is the least restrictive one and is sufficient to take the  $\omega \rightarrow \infty$  limit in a consistent manner, if one is only interested in geometrical and kinematical aspects. The first choice represents a stronger condition and should be adopted when one is also interested in taking the non-relativistic limit of dynamical aspects of general relativity. In particular, this constraint implies that  $M_\mu$  is pure gauge and does not represent extra degrees of freedom in the parent relativistic theory.

We can then define spin- and boost-connections  $\tilde{\omega}_\mu^{ab}(\tilde{e}, \tilde{\tau}, \tilde{m})$ ,  $\tilde{\omega}_\mu^a(\tilde{e}, \tilde{\tau}, \tilde{m})$ , that will be identified with the non-relativistic ones when  $\omega \rightarrow \infty$ , as the coefficients of the terms of leading order in an  $\omega$ -expansion of  $\Omega_\mu^{AB}(E)$ :

$$\Omega_\mu^{ab}(E) = \tilde{\omega}_\mu^{ab}(\tilde{e}, \tilde{\tau}, \tilde{m}) + \mathcal{O}\left(\frac{1}{\omega^2}\right), \quad (21)$$

$$\Omega_\mu^{0a}(E) = \frac{1}{\omega} \tilde{\omega}_\mu^a(\tilde{e}, \tilde{\tau}, \tilde{m}) + \mathcal{O}\left(\frac{1}{\omega^3}\right), \quad (22)$$

where

$$\tilde{\omega}_\mu{}^{ab}(\tilde{e}, \tilde{\tau}, \tilde{m}) = -2 \tilde{e}^\nu{}^{[a} \partial_{[\mu} \tilde{e}_{\nu]}{}^{b]} + \tilde{e}_\mu{}^c \tilde{e}^{\rho a} \tilde{e}^{\nu b} \partial_{[\rho} \tilde{e}_{\nu]}{}^c - \tilde{\tau}_\mu \tilde{e}^{\rho a} \tilde{e}^{\nu b} \partial_{[\rho} \tilde{m}_{\nu]}, \quad (23)$$

$$\tilde{\omega}_\mu{}^a(\tilde{e}, \tilde{\tau}, \tilde{m}) = \tilde{\tau}^\nu \partial_{[\mu} \tilde{e}_{\nu]}{}^a + \tilde{e}_\mu{}^b \tilde{e}^{\rho a} \tilde{\tau}^\nu \partial_{[\rho} \tilde{e}_{\nu]}{}^b + \tilde{e}^{\nu a} \partial_{[\mu} \tilde{m}_{\nu]} - \tilde{\tau}_\mu \tilde{e}^{\rho a} \tilde{\tau}^\nu \partial_{[\rho} \tilde{m}_{\nu]}. \quad (24)$$

Note that to obtain these formulas, one only needs the kinematical constraint (20) in the form

$$\omega \tilde{e}^\mu{}_a \tilde{e}^\nu{}_b \partial_{[\mu} \tilde{\tau}_{\nu]} = \frac{1}{2\omega} \tilde{e}^\mu{}_a \tilde{e}^\nu{}_b \partial_{[\mu} \tilde{m}_{\nu]}, \quad (25)$$

to replace terms that diverge in the  $\omega \rightarrow \infty$  limit by terms that have the correct leading  $\omega$ -order as indicated in the expansions (21) and (22). Since the stronger constraint (19) implies (25), it will achieve the same goal. The subleading terms in (21) and (22) are due to the fact that the relativistic spin-connection  $\Omega_\mu{}^{AB}(E)$  depends on the inverse vielbein  $E^\mu{}_A$ .

The rationale behind the redefinitions (15)–(22) is that they leave the sum of the products of the gauge-fields with their respective generators invariant, up to subleading terms in  $\omega$ , that stem from the dependent spin-connection via (21) and (22). One thus has:

$$\begin{aligned} & \hat{P}_A E^\mu{}_A + \mathcal{Z} M_\mu + M_{AB} \Omega_\mu{}^{AB}(E) \\ &= \tilde{P}_a \tilde{e}_\mu{}^a + \tilde{H} \tilde{\tau}_\mu + \tilde{Z} \tilde{m}_\mu + \tilde{J}_{ab} \tilde{\omega}_\mu{}^{ab}(\tilde{e}, \tilde{\tau}, \tilde{m}) \\ & - 2 \tilde{G}_a \tilde{\omega}_\mu{}^a(\tilde{e}, \tilde{\tau}, \tilde{m}) + \mathcal{O}\left(\frac{1}{\omega^2}\right). \end{aligned} \quad (26)$$

We proceed by taking the limit  $\omega \rightarrow \infty$  and derive the kinematics of Newton–Cartan gravity. For example, dropping the tildes on all fields we see that the expressions (23) and (24) coincide with the expressions of [23], which were obtained by setting

$$\begin{aligned} R_{\mu\nu}{}^a(P) &= 2 \partial_{[\mu} e_{\nu]}{}^a - 2 \omega_{[\mu}{}^{ab} e_{\nu]}{}^b - 2 \omega_{[\mu}{}^a \tau_{\nu]} = 0, \\ R_{\mu\nu}(Z) &= 2 \partial_{[\mu} m_{\nu]} - 2 \omega_{[\mu}{}^a e_{\nu]}{}^a = 0. \end{aligned} \quad (27)$$

These constraints are thus satisfied identically. Further constraints can be derived from the relativistic Bianchi identity (8) and the constraints on the curvature of the relativistic gauge-field  $M_\mu$ , see equation (19) or (20). Using the inverse vielbein (17) and the expansion

$$R_{\mu\nu}{}^{AB}(\Omega) = \delta_a^A \delta_b^B \tilde{R}_{\mu\nu}{}^{ab}(\tilde{J}) - \frac{1}{\omega} \delta_a^A \delta_0^B \tilde{R}_{\mu\nu}{}^a(\tilde{G}) + \frac{1}{\omega} \delta_0^A \delta_b^B \tilde{R}_{\mu\nu}{}^b(\tilde{G}), \quad (28)$$

in (8) we obtain the non-relativistic Bianchi identities

$$R_{[\mu\nu}{}^a(G) e_{\rho]}{}^a = 0, \quad R_{[\mu\nu}{}^{ab}(J) e_{\rho]}{}^b + R_{[\mu\nu}{}^a(G) \tau_{\rho]} = 0, \quad (29)$$

where the curvatures of the spin- and boost-connection gauge-fields are given by

$$\begin{aligned} R_{\mu\nu}{}^{ab}(J) &= 2 \partial_{[\mu} \omega_{\nu]}{}^{ab} - 2 \omega_{[\mu}{}^a{}_c \omega_{\nu]}{}^{cb}, \\ R_{\mu\nu}{}^a(G) &= 2 \partial_{[\mu} \omega_{\nu]}{}^a - 2 \omega_{[\mu}{}^{ab} \omega_{\nu]}{}^b. \end{aligned} \quad (30)$$

We consider the implications of equations (19) and (20) separately. In the first case (19), we simply find

$$R_{\mu\nu}(H) = 2 \partial_{[\mu} \tau_{\nu]} = 0. \quad (31)$$

In the kinematical case (20), we find the less restrictive condition

$$R_{ab}(H) = 2 e^\mu{}_a e^\nu{}_b \partial_{[\mu} \tau_{\nu]} = 0. \quad (32)$$

Interestingly, this constraint is equivalent to writing

$$\partial_{[\mu} \tau_{\nu]} = b_{[\mu} \tau_{\nu]}, \quad (33)$$

where  $b_\mu$  is completely arbitrary. This resembles the constraint found in the twistless torsional Newton–Cartan geometry of [25].

In either of the two cases, we can show that no further constraints are obtained by applying symmetry transformations on the constraints (31) and (32). To do so we need to derive the transformation rules of the Newton–Cartan fields  $\tau_\mu$ ,  $e_\mu{}^a$  and  $m_\mu$ . This can be done by applying the relativistic transformation rules (2) and (10) to the decompositions (15) and (18). For this purpose, we first express the new fields in terms of the old ones, i.e.

$$\tilde{\tau}_\mu = \frac{1}{2\omega} (E_\mu{}^0 + M_\mu), \quad \tilde{m}_\mu = \omega (E_\mu{}^0 - M_\mu). \quad (34)$$

Having done this, it is straightforward to obtain

$$\begin{aligned} \delta \tau_\mu &= 0, \\ \delta e_\mu{}^a &= \lambda^a{}_b e_\mu{}^b + \lambda^a \tau_\mu, \\ \delta m_\mu &= \partial_\mu \sigma + \lambda_a e_\mu{}^a, \end{aligned} \quad (35)$$

where we defined

$$\lambda^a = \omega \lambda^a{}_0, \quad \Lambda = -\frac{\sigma}{\omega}, \quad (36)$$

in agreement with equations (22) and (18). All fields transform under diffeomorphisms in the usual way. The transformations of the spin-connections can be found as well:

$$\begin{aligned} \delta \omega_\mu{}^{ab}(e, \tau, m) &= \partial_\mu \lambda^{ab} + 2 \lambda^{[a}{}_c \omega_\mu{}^{cb]}, \\ \delta \omega_\mu{}^a(e, \tau, m) &= \partial_\mu \lambda^a + \lambda^a{}_b \omega_\mu{}^b - \omega_\mu{}^a{}_c \lambda^c. \end{aligned} \quad (37)$$

The transformations (35)–(37) together with the constraints (31) and (29) make up the Newton–Cartan theory of gravity as described in [23].

At this point, we may impose equations of motion on the Newton–Cartan gauge-fields, in addition to the constraint (31) or (32), for example by performing the limiting procedure on the Einstein equations. One can show that this leads to the equation of motion presented in [23].

### 2.3. Example 2: three-dimensional on-shell Newton–Cartan supergravity

In this section we will extend the previous example to the three-dimensional Newton–Cartan supergravity theory constructed in [24]. The reason that we work in three-dimensions is that this is the only dimension in which an example of an on-shell Newton–Cartan supergravity theory is known so far. Since the discussion mainly parallels the previous section, we will skip most intermediate steps, where the contraction parameter  $\omega$  is finite, and we will mostly focus on the results obtained in the  $\omega \rightarrow \infty$  limit. Here and in the following, we will therefore no longer resort to the notation using tildes, to denote quantities at finite  $\omega$ .

The underlying gauge algebra in this case is the  $\mathcal{N} = 2$  Bargmann superalgebra. This superalgebra can be obtained by contracting the  $\mathcal{N} = 2$  Poincaré superalgebra, with central

extension  $\mathcal{Z}$ , that is given by

$$\begin{aligned} [M_{AB}, \hat{P}_C] &= -2 \eta_{C[A} \hat{P}_{B]}, & [M_{AB}, M_{CD}] &= 4 \eta_{[A[C} M_{D]B]}, \\ [M_{AB}, Q^i] &= -\frac{1}{2} \gamma_{AB} Q^i, & \{Q^i, Q^j\} &= -\gamma^A C^{-1} \hat{P}_A \delta^{ij} + C^{-1} \mathcal{Z} \epsilon^{ij}. \end{aligned} \quad (38)$$

Here, the supercharges  $Q^i$  ( $i = 1, 2$ ) are two-component Majorana spinors. For the gamma-matrices we choose a real basis, i.e.  $\gamma^A = (i\sigma_2, \sigma_1, \sigma_3)$  and the charge conjugation matrix is taken to be  $C = i\gamma^0$ . In order to define the Inönü–Wigner contraction, we first define the combinations

$$Q_{\pm} = \frac{1}{\sqrt{2}} (Q_1 \pm \gamma_0 Q_2), \quad (39)$$

and split the three-dimensional flat indices  $A, B$  into time-like and space-like indices  $\{0, a\}$ . As before, we set  $M_{ab} = J_{ab}$  for the purely spatial rotations. The motivation for choosing the combinations of the relativistic spinors as given in equation (39), stems from the non-relativistic algebra (and later on from the transformation rules of the gravitini). It leads to particularly simple transformations of the spinors under boosts. Before making the contraction we perform the following redefinition of the generators:

$$\begin{aligned} Q_- &\rightarrow \sqrt{\omega} Q_-, & Q_+ &\rightarrow \frac{1}{\sqrt{\omega}} Q_+, & M_{a0} &\rightarrow \omega G_a, \\ \mathcal{Z} &\rightarrow -\omega Z + \frac{1}{2\omega} H, & \hat{P}_0 &\rightarrow \omega Z + \frac{1}{2\omega} H. \end{aligned} \quad (40)$$

Using these redefinitions, the supersymmetric extension of the Bargmann algebra is then obtained in the limit  $\omega \rightarrow \infty$ , in a similar way as discussed in the previous subsection. In particular, we find the following non-vanishing commutation relations:

$$\begin{aligned} [J_{ab}, P_c] &= -2 \delta_{c[a} P_{b]}, & [J_{ab}, G_c] &= -2 \delta_{c[a} G_{b]}, \\ [G_a, H] &= -P_a, & [G_a, P_b] &= -\delta_{ab} Z, \\ [J_{ab}, Q_{\pm}] &= -\frac{1}{2} \gamma_{ab} Q_{\pm}, & [G_a, Q_+] &= -\frac{1}{2} \gamma_{a0} Q_-, \\ \{Q_+, Q_+\} &= -\gamma^0 C^{-1} H, & \{Q_+, Q_-\} &= -\gamma^a C^{-1} P_a, \\ \{Q_-, Q_-\} &= -2 \gamma^0 C^{-1} Z. \end{aligned} \quad (41)$$

The bosonic part of the algebra corresponds to the Bargmann algebra, see equation (14). Note that, since we are working in three-dimensions, the spatial rotations are abelian.

We now wish to extend this contraction to the fields of the on-shell, relativistic  $\mathcal{N} = 2$  supergravity multiplet, whose supersymmetry transformation rules (with parameter  $\eta_i$ ) are given by

$$\delta E_{\mu}{}^A = \frac{1}{2} \delta^{ij} \bar{\eta}_i \gamma^A \Psi_{\mu j}, \quad (42)$$

$$\delta \Psi_{\mu i} = D_{\mu} \eta_i = \partial_{\mu} \eta_i - \frac{1}{4} \Omega_{\mu}{}^{AB} (E, \Psi_i) \gamma_{AB} \eta_i, \quad (43)$$

where  $D_\mu$  is the Lorentz-covariant derivative and the dependent spin-connection  $\Omega_\mu^{AB}(E, \Psi_i)$  is given by the supersymmetric analog of (7), i.e.

$$\begin{aligned} \Omega_\mu^{AB}(E, \Psi_i) = & -2 E^{\rho[A} \left( \partial_{[\mu} E_{\rho]}^{B]} - \frac{1}{4} \delta^{ij} \bar{\Psi}_{[\mu i} \gamma^{B]} \Psi_{\nu] j} \right) \\ & + E_{\mu C} E^{\rho A} E^{\nu B} \left( \partial_{[\rho} E_{\nu]}^C - \frac{1}{4} \delta^{ij} \bar{\Psi}_{[\rho i} \gamma^C \Psi_{\nu] j} \right). \end{aligned} \quad (44)$$

From this expression one derives that the supersymmetry transformation of the (dependent) spin-connection is given by

$$\delta \Omega_\mu^{AB}(E, \Psi_i) = -\frac{1}{2} \delta^{ij} E^{\rho[A} \bar{\eta}_i \gamma^{B]} \hat{\Psi}_{\mu \rho j} + \frac{1}{4} \delta^{ij} E_{\mu C} E^{\rho A} E^{\nu B} \bar{\eta}_i \gamma^C \hat{\Psi}_{\rho \nu j}. \quad (45)$$

Note that this transformation rule is zero on-shell, i.e. it vanishes upon using the fermionic equations of motion

$$\hat{\Psi}_{\mu \nu i} = 2 D_{[\mu} \Psi_{\nu] i} = 0. \quad (46)$$

One may verify that the supersymmetry algebra on the fields (42) and (43) closes on-shell.

As in the previous subsection, we will introduce a field  $M_\mu$ , associated to the central charge transformation  $\mathcal{Z}$  of the  $\mathcal{N} = 2$  algebra. Its transformation rule under supersymmetry is determined by the Poincaré superalgebra (38)

$$\delta M_\mu = \frac{1}{2} \varepsilon^{ij} \bar{\eta}_i \Psi_{\mu j}. \quad (47)$$

This field is ordinarily not introduced in the supergravity multiplet. In order not to upset the on-shell counting of bosonic and fermionic degrees of freedom, we are thus obliged to set the supercovariant curvature of  $M_\mu$  to zero, i.e.

$$\hat{F}_{\mu\nu}(M) = 2 \partial_{[\mu} M_{\nu]} - \frac{1}{2} \varepsilon^{ij} \bar{\Psi}_{[\mu i} \Psi_{\nu] j} = 0, \quad (48)$$

so that this field corresponds to a pure gauge degree of freedom. Note that this constraint also implies that the commutator of two supersymmetry transformations acting on  $M_\mu$  closes to a general coordinate transformation and a central charge transformation. Moreover, since this constraint is the supercovariant version of (19), it will allow us to obtain finite expressions for the non-relativistic spin-connections from the relativistic one. Starting from expression (48), the full set of relativistic equations of motion is obtained by the following chain of supersymmetry transformations

$$\hat{F}_{\mu\nu}(M) = 0 \quad \rightarrow \quad \hat{\Psi}_{\mu \nu i} = 0 \quad \rightarrow \quad \hat{R}_{\mu\nu}^{AB}(\Omega) = 0. \quad (49)$$

This concludes the summary of our relativistic starting point. Let us now extend the algebra contraction to the fields of this on-shell supergravity multiplet. For the bosonic fields, this entails the redefinitions involving  $\omega$  that were introduced in the previous section. The redefinitions of the gravitini follow from the way we contract the generators of the three-dimensional  $\mathcal{N} = 2$  Poincaré superalgebra to get the Bargmann superalgebra, see the definitions (39). Hence, we define new spinors

$$\Psi_\pm = \frac{1}{\sqrt{2}} (\Psi_1 \pm \gamma_0 \Psi_2), \quad (50)$$

and similarly for the parameters  $\eta$ . We then introduce the scalings:

$$\begin{aligned}\Psi_{\mu+} &= \sqrt{\omega} \psi_{\mu+}, & \eta_+ &= \sqrt{\omega} \epsilon_+, \\ \Psi_{\mu-} &= \frac{1}{\sqrt{\omega}} \psi_{\mu-}, & \eta_- &= \frac{1}{\sqrt{\omega}} \epsilon_-.\end{aligned}\quad (51)$$

The following non-relativistic supersymmetry transformation rules then follow

$$\begin{aligned}\delta\tau_\mu &= \frac{1}{2} \bar{\epsilon}_+ \gamma^0 \psi_{\mu+}, \\ \delta e_\mu^a &= \frac{1}{2} \bar{\epsilon}_+ \gamma^a \psi_{\mu-} + \frac{1}{2} \bar{\epsilon}_- \gamma^a \psi_{\mu+}, \\ \delta m_\mu &= \bar{\epsilon}_- \gamma^0 \psi_{\mu-},\end{aligned}\quad (52)$$

as well as

$$\begin{aligned}\delta\psi_{\mu+} &= \partial_\mu \epsilon_+ - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \epsilon_+, \\ \delta\psi_{\mu-} &= \partial_\mu \epsilon_- - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \epsilon_- + \frac{1}{2} \omega_\mu^a \gamma_{a0} \epsilon_+.\end{aligned}\quad (53)$$

The transformation rules of the spinors with respect to the non-relativistic bosonic symmetries stem from the relativistic rule  $\delta\Psi_\mu = 1/4 \lambda^{AB} \gamma_{AB} \Psi_\mu$  and are found to be

$$\begin{aligned}\delta\psi_{\mu+} &= \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{\mu+}, \\ \delta\psi_{\mu-} &= \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{\mu-} - \frac{1}{2} \lambda^a \gamma_{a0} \psi_{\mu+}.\end{aligned}\quad (54)$$

It is understood that the spin-connections  $\omega_\mu^a, \omega_\mu^{ab}$  in (53) are dependent, i.e.  $\omega_\mu^a = \omega_\mu^a(e, \tau, m, \psi_\pm)$  and  $\omega_\mu^{ab} = \omega_\mu^{ab}(e, \tau, m, \psi_\pm)$ . The expressions for these non-relativistic spin-connections can be obtained from the relativistic expressions given in equations (44) and (48). We find

$$\begin{aligned}\omega_\mu^{ab}(e, \tau, m, \psi_\pm) &= -2 e^\nu{}^{[a} \left( \partial_{[\mu} e_{\nu]}{}^{b]} - \frac{1}{2} \bar{\psi}_{[\mu+} \gamma^{b]} \psi_{\nu]-} \right) \\ &\quad + e_\mu{}^c e^{\rho a} e^{\nu b} \left( \partial_{[\rho} e_{\nu]}{}^c - \frac{1}{2} \bar{\psi}_{[\rho+} \gamma^c \psi_{\nu]-} \right) \\ &\quad - \tau_\mu e^{\rho a} e^{\nu b} \left( \partial_{[\rho} m_{\nu]} - \frac{1}{2} \bar{\psi}_{[\rho-} \gamma^0 \psi_{\nu]-} \right),\end{aligned}\quad (55)$$

$$\begin{aligned}\omega_\mu^a(e, \tau, m, \psi_\pm) &= \tau^\nu \left( \partial_{[\mu} e_{\nu]}{}^a - \frac{1}{2} \bar{\psi}_{[\mu+} \gamma^a \psi_{\nu]-} \right) \\ &\quad + e_{\mu b} e^{\rho a} \tau^\nu \left( \partial_{[\rho} e_{\nu]}{}^b - \frac{1}{2} \bar{\psi}_{[\rho+} \gamma^b \psi_{\nu]-} \right) \\ &\quad + e^{\nu a} \left( \partial_{[\mu} m_{\nu]} - \frac{1}{2} \bar{\psi}_{[\mu-} \gamma^0 \psi_{\nu]-} \right) \\ &\quad - \tau_\mu e^{\rho a} \tau^\nu \left( \partial_{[\rho} m_{\nu]} - \frac{1}{2} \bar{\psi}_{[\rho-} \gamma^0 \psi_{\nu]-} \right).\end{aligned}\quad (56)$$

In order to obtain these expression, we have mimicked the discussion around equations (21)–(25). This time however, we have used equation (48) to replace terms that diverge in the



$\omega \rightarrow \infty$  limit, by terms with the expected  $\omega$ -order. Like in the bosonic case, the above expressions for the spin-connections identically solve the supercovariant curvature constraints

$$\begin{aligned}\hat{R}_{\mu\nu}{}^a(P) &= R_{\mu\nu}{}^a(P) - \bar{\psi}_{[\mu+\gamma^a\psi_{\nu]-} = 0, \\ \hat{R}_{\mu\nu}(Z) &= R_{\mu\nu}(Z) - \bar{\psi}_{[\mu-\gamma^0\psi_{\nu]-} = 0.\end{aligned}\quad (57)$$

The so-called conventional constraints (57) are identically fulfilled, so we need not worry about variations thereof. They can be used to determine the transformations of the spin- and boost-connections (55) and (56).

The  $\omega \rightarrow \infty$  limit of (48) leads to the further constraint

$$\hat{R}_{\mu\nu}(H) = R_{\mu\nu}(H) - \frac{1}{2} \bar{\psi}_{[\mu+\gamma^0\psi_{\nu]+} = 0. \quad (58)$$

This leads to further conditions upon variation under supersymmetry. To check this we need to know the supersymmetry variations of the spin-connections. Using the transformation rules (52) and (53) in the expressions (55), (56) we find

$$\begin{aligned}\delta_Q \omega_\mu{}^{ab}(e, \tau, m, \psi_\pm) &= \frac{1}{2} \bar{\epsilon}_+\gamma^{[b}\hat{\psi}^{a]}_{\mu-} + \frac{1}{4} e_{\mu c} \bar{\epsilon}_+\gamma^c\hat{\psi}^{ab}_{-} - \frac{1}{2} \tau_\mu \bar{\epsilon}_-\gamma^0\hat{\psi}^{ab}_{-} \\ &\quad + \frac{1}{2} \bar{\epsilon}_-\gamma^{[b}\hat{\psi}^{a]}_{\mu+} + \frac{1}{4} e_{\mu c} \bar{\epsilon}_-\gamma^c\hat{\psi}^{ab}_{+},\end{aligned}\quad (59)$$

$$\begin{aligned}\delta_Q \omega_\mu{}^a(e, \tau, m, \psi_\pm) &= \frac{1}{2} \bar{\epsilon}_-\gamma^0\hat{\psi}_\mu{}^a_{-} + \frac{1}{2} \tau_\mu \bar{\epsilon}_-\gamma^0\hat{\psi}_0{}^a_{-} + \frac{1}{4} e_{\mu b} \bar{\epsilon}_+\gamma^b\hat{\psi}_0{}^a_{-} + \frac{1}{4} \bar{\epsilon}_+\gamma^a\hat{\psi}_{\mu 0-} \\ &\quad + \frac{1}{4} e_{\mu b} \bar{\epsilon}_-\gamma^b\hat{\psi}_0{}^a_{+} + \frac{1}{4} \bar{\epsilon}_-\gamma^a\hat{\psi}_{\mu 0+}.\end{aligned}\quad (60)$$

Now we readily derive that under supersymmetry transformations, with parameters  $\epsilon_+$  and  $\epsilon_-$ , the following set of constraints is generated:

$$\xrightarrow{Q_-} \hat{\psi}_{ab-} = 0, \quad (61)$$

$$\hat{R}_{\mu\nu}(H) = 0 \xrightarrow{Q_+} \hat{\psi}_{\mu\nu+} = 0, \quad \xrightarrow{Q_+} R_{\mu\nu}{}^{ab}(J) = 0. \quad (62)$$

$$\xrightarrow{Q_+} \gamma^a\hat{\psi}_{a0-} = 0 \xrightarrow{Q_+} \hat{R}_{0a}{}^a(G) = 0 \quad (63)$$

Note that the variation of the  $\hat{\psi}_{\mu\nu+} = 0$  constraint leads to three different constraints. Of these three constraints only the variation of the  $\gamma^a\hat{\psi}_{a0-} = 0$  leads to one more constraint. Using the last constraint given in equation (62) the non-relativistic Bianchi identities reduce to

$$\hat{R}_{ab}{}^c(G) = 0, \quad \hat{R}_{0[a}{}^{b]}(G) = 0. \quad (64)$$

These identities are e.g. needed to show that the variation of the constraint given in equation (61) does not lead to further constraints. We did not check the variation of the last constraint in equation (63). The calculation is quite involved and has also not been carried out in [24]. We will, however, show in section 3 that the full set of constraints (61)–(63) can be derived from an off-shell version of this multiplet where we have checked the consistency of the whole set of constraints.

At this point we have finished the derivation of the three-dimensional on-shell Newton–Cartan supergravity constructed in [24], i.e. we obtained all constraints and transformation rules. The terminology ‘on-shell’ stems from the fact that the constraints given in

$$\begin{array}{ccc}
\hat{F}_{\mu\nu}(M) = 0 & \xrightarrow{\omega \rightarrow \infty} & \hat{R}_{\mu\nu}(H) = 0 \\
\downarrow Q_i & & \downarrow Q_+ \\
\hat{\Psi}_{\mu\nu i} = 0 & \xrightarrow{\omega \rightarrow \infty} & \hat{\psi}_{\mu\nu+} = 0 \\
\downarrow Q_i & & \downarrow Q_+ \\
R_{\mu\nu}{}^{AB}(\Omega) = 0 & \xrightarrow{\omega \rightarrow \infty} & R_{\mu\nu}{}^{ab}(J) = 0
\end{array}$$

**Figure 1.** The chains of constraints in the relativistic (left column) and non-relativistic (right column) case. In the non-relativistic case we do not denote the complete chain, as given in equations (61)–(63).

equation (63) both can be interpreted as equations of motion for Newton–Cartan supergravity: the first constraint is necessary to obtain closure of the supersymmetry algebra while the bosonic part of the second constraint is precisely the equation of motion of the bosonic Newton–Cartan gravity theory. Note, however, that to call some constraints ‘equation of motion’ and others not is slightly ambiguous when talking about Newton–Cartan (super) gravity, due to the absence of an action principle that can be used to derive these equations of motion. In section 3, we will construct a different, ‘off-shell’ version of three-dimensional Newton–Cartan supergravity, that includes an auxiliary scalar field in the supermultiplet. The terminology ‘off-shell’ will be justified in the sense that the first constraint given in equation (63) will no longer be needed for closure of the supersymmetry algebra. Both constraints given in equation (63) will in fact not appear at all. Equations of motion can thus be identified in a pragmatic way as those constraints that can be removed by adding auxiliary degrees of freedom to a non-relativistic supermultiplet.

Let us stress/repeat some important aspects of this second example. In the case at hand we can draw the commuting diagram, given in figure 1. The left column represents a chain of relativistic constraints while the right column contains a similar chain of non-relativistic constraints.

The diagram shows all non-relativistic constraints that are obtained by a  $\omega \rightarrow \infty$  limit of the relativistic ones. However, in this way we do not obtain the full set of non-relativistic constraints. This is due to the fact that in the left column we have included both supersymmetries but in the right column we have only included the variation under  $Q_+$  transformations. Further constraints follow from the variation under the  $Q_-$  transformations, but those non-relativistic constraints are not obtained as limits of relativistic constraints.

We note that in the limit  $\omega \rightarrow \infty$  the two relativistic constraints given in the second row of the left column, namely those containing the two gravitino curvatures, lead to just the single non-relativistic constraint given in the second row of the right column. This is in line with the fact that the constraint  $\hat{R}_{\mu\nu}(H) = 0$  only varies under one of the two non-relativistic supersymmetries and hence its variation under supersymmetry only leads to one of the non-relativistic gravitino curvatures. This observation is of vital importance to understand the off-shell case treated in section 3. There we are also going to impose the constraint  $\hat{F}_{\mu\nu}(M) = 0$ , but since its non-relativistic limit does not necessarily lead to the non-relativistic equations of motion, imposing this constraint does not force us to go to the non-relativistic on-shell multiplet of the current section.

We finish this section with a third illustration of the non-relativistic limiting procedure in which we consider a superparticle in a curved background.

#### 2.4. Example 3: the non-relativistic superparticle in a curved background

In this third example we apply the limiting procedure to a superparticle moving in a curved background. To be concrete, we use it to derive the action and transformation rules of the non-relativistic superparticle in a curved background, put forward in [30]. The non-relativistic superparticle in a flat background was already discussed in [35–37]. We note that the limit that was taken in [34] to derive the non-relativistic superparticle in a flat background can be understood as a special case of the analysis in this section.

It is illustrative to first discuss the bosonic particle. To derive the action of a non-relativistic bosonic point-particle in an arbitrary Newton–Cartan background we start from the relativistic action

$$S_{\text{rel}} = -M \int d\lambda \left( \sqrt{-\eta_{AB} (\dot{x}^\mu E_\mu^A) (\dot{x}^\nu E_\nu^B)} - \dot{x}^\mu M_\mu \right). \quad (65)$$

All dots refer to derivatives w.r.t. the worldline parameter  $\lambda$ , i.e.  $\dot{x}^\mu = dx^\mu/d\lambda$ . We use mostly plus signature and we also added a ‘charge’ term  $\dot{x}^\mu M_\mu$ . Here, we impose that the curvature of the abelian gauge-field  $M_\mu$  vanishes, implying that it can locally be written as  $M_\mu = \partial_\mu \Gamma$  and the second term in (65) corresponds to a total derivative. Using the expressions (15) and (18) in the relativistic action (65) and taking  $M = \omega m$ , we obtain, in the limit  $\omega \rightarrow \infty$ , the following non-relativistic action:

$$S_{\text{nr}} = m \int d\lambda \left[ \frac{\delta_{ab} (\dot{x}^\mu e_\mu^a) (\dot{x}^\nu e_\nu^b)}{2\tau_\rho \dot{x}^\rho} - m_\mu \dot{x}^\mu \right]. \quad (66)$$

This action agrees with the action, calculated by other means, in e.g. [4, 28, 38]. Note that one of the reasons to add the term  $\dot{x}^\mu M_\mu$  is to cancel a divergent (total derivative) term that otherwise would arise in the limit  $\omega \rightarrow \infty$ , see also [32]. In contrast, the combination  $\dot{x}^\mu m_\mu$  in the non-relativistic action is not a total derivative term. This non-relativistic term does not only follow from the relativistic  $\dot{x}^\mu M_\mu$  term, but it also receives contributions from the kinematic term  $\sqrt{-\dot{x}^2}$ .

We now generalize the discussion of the non-relativistic bosonic particle to the non-relativistic superparticle. The relativistic superparticle in a curved background is most conveniently written using superspace techniques, see [39]. Since so far a non-relativistic superspace description is lacking, we will refrain from using superspace notation and simplify the discussion and notation by considering only the terms in the action that are at most quadratic in the fermions. Thus, the supersymmetric analog of (65) takes the form

$$S_{\text{rel}} = -M \int d\lambda \left[ \sqrt{-\eta_{AB} \Pi^A \Pi^B} - \frac{1}{4} \varepsilon^{ij} \bar{\theta}_i D_\lambda \theta_j - \dot{x}^\mu \left( M_\mu - \frac{1}{2} \varepsilon^{ij} \bar{\theta}_i \Psi_{ij} \right) \right]. \quad (67)$$

The background fields  $E_\mu^A$ ,  $M_\mu$  and  $\Psi_{ij}$  are those of the three-dimensional on-shell theory discussed in section 2.3 and the embedding coordinates are  $x^\mu$  and  $\theta_i$ . The supersymmetric line-element  $\Pi^A$  is defined as

$$\Pi^A = \dot{x}^\mu \left( E_\mu^A - \frac{1}{2} \delta^{ij} \bar{\theta}_i \gamma^A \Psi_{ij} \right) + \frac{1}{4} \delta^{ij} \bar{\theta}_i \gamma^A D_\lambda \theta_j, \quad (68)$$

where the derivative  $D_\lambda$  is covariantized w.r.t. Lorentz transformations, i.e.

$$D_\lambda \theta = \dot{\theta} - \frac{1}{4} \dot{x}^\mu \Omega_\mu^{AB}(E, \Psi_i) \gamma_{AB} \theta. \quad (69)$$

As we are only interested in terms up to second order in fermions expressions like  $\eta_{AB} \Pi^A \Pi^B$  are understood to contain only such terms and all terms quartic in fermions are discarded.

The action (67) is invariant under the following supersymmetry transformations of the embedding coordinates

$$\delta x^\mu = -\frac{1}{4} \delta^{ij} \bar{\eta}_i \gamma^A \theta_j E^\mu{}_A, \quad \delta \theta_i = \eta_i. \quad (70)$$

These transformations should be accompanied by the following  $\sigma$ -model transformations [40, 41] of the background fields, as explained e.g. in [4, 30]:

$$\begin{aligned} \delta E_\mu{}^A &= \frac{1}{2} \delta^{ij} \bar{\eta}_i \gamma^A \Psi_{uj} - \frac{1}{4} \delta^{ij} \bar{\eta}_i \gamma^B \theta_j E^\rho{}_B \partial_\rho E_\mu{}^A, \quad \delta \Psi_{\mu i} = D_\mu \eta_i, \\ \delta M_\mu &= \frac{1}{2} \varepsilon^{ij} \bar{\eta}_i \Psi_{uj} - \frac{1}{4} \delta^{ij} \bar{\eta}_i \gamma^B \theta_j E^\rho{}_B \partial_\rho M_\mu. \end{aligned} \quad (71)$$

The action (67) is also left invariant by the  $\kappa$ -transformations

$$\delta_\kappa x^\mu = -\frac{1}{4} \delta^{ij} \bar{\theta}_i \gamma^A \delta_\kappa \theta_j E^\mu{}_A, \quad \delta_\kappa \theta_1 = \kappa, \quad \delta_\kappa \theta_2 = -\frac{\Pi^A \gamma_A}{\sqrt{-\Pi^2}} \kappa. \quad (72)$$

In this case all background fields transform under  $\kappa$ -symmetry only through their dependence on the embedding coordinates. To show invariance under supersymmetry and  $\kappa$ -symmetry, one needs to use the equations of motion of the background fields.

With all these preliminaries at hand it is now straightforward to apply the limiting procedure to the relativistic superparticle action (67). This yields the following result:

$$\begin{aligned} S_{\text{nr}} &= \frac{m}{2} \int d\lambda \left[ \frac{\hat{\pi}^a \hat{\pi}^b \delta_{ab}}{\hat{\pi}^0} - 2 \dot{x}^\mu (m_\mu - \bar{\theta}_- \gamma^0 \psi_{\mu-}) \right. \\ &\quad \left. - \bar{\theta}_- \gamma^0 \hat{D} \theta_- - \frac{1}{2} \dot{x}^\mu \omega_\mu{}^a \bar{\theta}_+ \gamma_a \theta_- \right], \end{aligned} \quad (73)$$

where we have defined the following supersymmetric line elements

$$\hat{\pi}^0 = \dot{x}^\mu \left( \tau_\mu - \frac{1}{2} \bar{\theta}_+ \gamma^0 \psi_{\mu+} \right) + \frac{1}{4} \bar{\theta}_+ \gamma^0 \hat{D} \theta_+, \quad (74)$$

$$\begin{aligned} \hat{\pi}^a &= \dot{x}^\mu \left( e_\mu{}^a - \frac{1}{2} \bar{\theta}_+ \gamma^a \psi_{\mu-} - \frac{1}{2} \bar{\theta}_- \gamma^a \psi_{\mu+} \right) + \frac{1}{4} \bar{\theta}_+ \gamma^a \hat{D} \theta_- + \frac{1}{4} \bar{\theta}_- \gamma^a \hat{D} \theta_+ \\ &\quad + \frac{1}{8} \bar{\theta}_+ \gamma^a \gamma_{b0} \theta_+ \dot{x}^\mu \omega_\mu{}^b. \end{aligned} \quad (75)$$

Note that the supercovariant derivative  $\hat{D}$  is covariant w.r.t. spatial rotations, not boosts. The boost-connection  $\omega_\mu{}^a$  that appears in equations (73) and (75) is the dependent boost-connection (56). For notational simplicity we do not denote below its dependence on the other fields. The transformations of the embedding coordinates under  $\kappa$ -symmetry are given by

$$\begin{aligned}\delta t &= -\frac{1}{4} \bar{\theta}_+ \gamma^0 \kappa, & \delta \theta_+ &= \kappa, \\ \delta x^i &= -\frac{1}{4} \bar{\theta}_- \gamma^i \kappa - \frac{1}{8} \frac{\hat{\pi}^k}{\hat{\pi}^0} \bar{\theta}_+ \gamma^{0i} \gamma_k \kappa, & \delta \theta_- &= -\frac{\hat{\pi}^i}{2\hat{\pi}^0} \gamma_{i0} \kappa.\end{aligned}\quad (76)$$

This reproduces precisely, to second order in fermions, the  $\kappa$ -symmetric non-relativistic superparticle in a curved background as presented in [30].

Fixing kappa-symmetry by setting  $\theta_+ = 0$  we obtain the result of [30]. When we gauge-fix the Newton–Cartan background to a Galilean background with a Newton potential  $\Phi$ , the one described in [24], the action (73) reduces to

$$S_{\text{nr}} = \frac{m}{2} \int d\lambda \left[ \frac{\pi_\Phi^i \pi_\Phi^i}{\pi^0} - 2i(\Phi - \bar{\theta}_- \gamma^0 \Psi) - \bar{\theta}_- \gamma^0 \dot{\theta}_- + \frac{i}{2} \partial_i \Phi \bar{\theta}_+ \gamma^i \dot{\theta}_- \right], \quad (77)$$

with the ‘super-Galilean’ line-elements given by

$$\pi^0 = t + \frac{1}{4} \bar{\theta}_+ \gamma^0 \dot{\theta}_+, \quad (78)$$

$$\pi_\Phi^i = \dot{x}^i - \frac{1}{2} i \bar{\theta}_+ \gamma^i \Psi + \frac{1}{4} \bar{\theta}_+ \gamma^i \dot{\theta}_- + \frac{1}{4} \bar{\theta}_- \gamma^i \dot{\theta}_+ - \frac{1}{8} i \partial_j \Phi \bar{\theta}_+ \gamma^i \gamma_{j0} \theta_+. \quad (79)$$

This finishes our discussion of the superparticle in a non-relativistic curved background.

### 3. 3D non-relativistic off-shell supergravity

In this section, we apply the limiting procedure to obtain an *off-shell* version of the Newton–Cartan supergravity theory of [24], that was revisited in section 2.3. Such an off-shell version will necessarily contain auxiliary fields, that cannot be interpreted as gauge-fields of an underlying symmetry algebra. One can therefore not use a gauging procedure to obtain this theory. This example shows that the limiting procedure provides us with an effective tool to obtain non-relativistic theories from relativistic ones, when no gauging procedure is available.

We will start from an off-shell formulation of three-dimensional  $\mathcal{N} = 2$  supergravity. There exist two different such off-shell formulations [42–44]. We will start from the so-called three-dimensional new minimal  $\mathcal{N} = 2$  Poincaré multiplet, since this multiplet contains the abelian central charge gauge-field  $M_\mu$  that was already needed in the on-shell case. Due to the lack of a vector gauge-field with the transformation rule given by equation (47) it is not obvious how to take the non-relativistic limit of the old minimal  $\mathcal{N} = 2$  Poincaré multiplet.

The new minimal multiplet consists of a dreibein  $E_\mu^A$ , two gravitini  $\Psi_{\mu i}$  ( $i = 1, 2$ ), two auxiliary vector gauge-fields  $M_\mu$  and  $V_\mu$  and an auxiliary scalar  $D$ , see e.g. [44]. The supersymmetries (with parameters  $\eta_i$ ), the central charge transformations (with parameter  $\Lambda$ ) and the  $R$ -symmetry transformations (with parameter  $\rho$ ) of these fields are given by

$$\begin{aligned}\delta E_\mu^A &= \frac{1}{2} \delta^{ij} \bar{\eta}_i \gamma^A \Psi_{\mu j}, \\ \delta \Psi_{\mu i} &= D_\mu \eta_i + \varepsilon^{ij} \eta_j V_\mu - \gamma_\mu \eta_i D + \frac{1}{4} \gamma_\mu \gamma \cdot \hat{F}(M) \varepsilon^{ij} \eta_j - \varepsilon^{ij} \Psi_{\mu j} \rho,\end{aligned}$$

$$\begin{aligned}
\delta M_\mu &= \frac{1}{2} \varepsilon^{ij} \bar{\eta}_i \Psi_{\mu j} + \partial_\mu \Lambda, \\
\delta V_\mu &= \frac{1}{2} \varepsilon^{ij} \bar{\eta}_i \gamma^\nu \hat{\Psi}_{\mu\nu j} - \frac{1}{8} \varepsilon^{ij} \bar{\eta}_i \gamma_\mu \gamma \cdot \hat{\Psi}_j \\
&\quad - \frac{1}{4} \delta^{ij} \bar{\eta}_i \gamma \cdot \hat{F}(M) \Psi_{\mu j} - \varepsilon^{ij} \bar{\eta}_i \Psi_{\mu j} D + \partial_\mu \rho, \\
\delta D &= -\frac{1}{16} \delta^{ij} \bar{\eta}_i \gamma \cdot \hat{\Psi}_j.
\end{aligned} \tag{80}$$

The field strength  $\hat{F}_{\mu\nu}(M)$  of the central charge gauge field is given by equation (48) while the two gravitino curvatures read

$$\hat{\Psi}_{\mu\nu i} = 2 D_{[\mu} \Psi_{\nu]i} - 2 \gamma_{[\mu} \Psi_{\nu]i} D - 2 \varepsilon^{ij} \Psi_{[\mu j} V_{\nu]} + \frac{1}{2} \varepsilon^{ij} \gamma_{[\mu} \gamma \cdot \hat{F}(M) \Psi_{\nu]j}. \tag{81}$$

The dots refer to gamma traces as in  $\gamma \cdot \hat{F}(M) = \gamma^{\mu\nu} \hat{F}_{\mu\nu}(M)$ . The spin-connection is determined by requiring that the supercovariant torsion  $\hat{R}_{\mu\nu}{}^A(E)$  is zero. Its supersymmetry variation follows from the expression in terms of  $E_\mu{}^A$  and  $\Psi_{\mu i}$ , see equation (44).

In order to apply the limiting procedure to the transformation rules given in equation (80), we use the same rescalings as in the previous sections, supplemented with

$$D \rightarrow \frac{1}{\omega} S. \tag{82}$$

We do not rescale the auxiliary field  $V_\mu$ . Below we will argue that in the non-relativistic limit one must eliminate  $V_\mu$ . The action of  $\mathcal{N} = 2$  new minimal supergravity contains a  $D^2$  term that plays the role of the cosmological constant  $\Lambda_{\text{CC}}$ . It is thus not surprising that in the non-relativistic limit,  $D$  scales like the square root of  $\Lambda_{\text{CC}}$ , see e.g. [30] for the non-relativistic contraction of the anti de Sitter algebra.

Going through similar arguments as in sections 2.2 and 2.3 we determine the non-relativistic spin-connections  $\omega_\mu{}^{ab}(e, \tau, m, \psi_\pm)$  and  $\omega_\mu{}^a(e, \tau, m, \psi_\pm)$  to be given by equations (55) and (56). As in the on-shell case, we need to impose equation (48) as an extra constraint, in order to take the non-relativistic limit consistently. In fact, to get the correct expressions for the spin-connections we only need to set to zero the spatial components of  $\hat{F}_{\mu\nu}(M)$ . However, this is not sufficient to take the non-relativistic limit in the transformation rules of all fields. Indeed, we need to eliminate also all remaining components of  $\hat{F}_{\mu\nu}(M)$  as well as of  $\hat{\Psi}_{\mu\nu+}$  to avoid divergent terms in the transformation rules of the fields. Since the relativistic constraint  $\hat{F}_{\mu\nu}(M) = 0$  varies under supersymmetry to the fermionic equations of motion, we effectively put the relativistic theory on-shell. In the following, we will show that, upon elimination of only one of the auxiliary fields, the limiting procedure leads to an irreducible non-relativistic multiplet on which the Bargmann superalgebra is realized off-shell, in a sense that we will clarify below.

Here, we present a brief discussion to argue why we can eliminate the auxiliary field  $V_\mu$ . In a first approach the limiting procedure leads to the constraints

$$\hat{R}_{\mu\nu}(H) = 0, \quad \hat{\psi}_{\mu\nu+} = 0, \quad \hat{\psi}_{ab-} = 0. \tag{83}$$

At this point we derive the following transformation rules for  $\tau_\mu$  and the auxiliary fields  $V_\mu$  and  $S$ :

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2} \bar{\epsilon}_+ \gamma^0 \psi_{\mu+}, \\
\delta V_\mu &= -\frac{1}{4} \bar{\epsilon}_+ \gamma^{a0} \hat{\psi}_{\mu a-} - \bar{\epsilon}_+ \gamma^0 \psi_{\mu+} S, \\
\delta S &= -\frac{1}{8} \bar{\epsilon}_+ \gamma^{a0} \hat{\psi}_{a0-}.
\end{aligned} \tag{84}$$

The supersymmetry transformations of the last two constraints of (83) imply

$$e^\mu{}_a e^\nu{}_b \hat{V}_{\mu\nu} = \hat{V}_{ab} = 0, \tag{85}$$

which is the spatial part of the supercovariant curvature of  $V_\mu$ :

$$\hat{V}_{\mu\nu} = 2 \partial_{[\mu} V_{\nu]} + \frac{1}{2} \bar{\psi}_{[\mu+} \gamma^{a0} \hat{\psi}_{\nu]a-} + \bar{\psi}_{[\mu+} \gamma^0 \psi_{\nu]+} S. \tag{86}$$

Using the first and last constraint in equation (83) we observe that the constraint (85) is always satisfied if we set

$$V_\mu = -2 \tau_\mu S. \tag{87}$$

The inverse vielbeins in (85) eliminate any term with a free  $\tau_\mu$  and thus the derivative in (86) must hit the  $\tau_\mu$  when we insert equation (87) in expression (86). We can then use the first constraint of (83) to cancel all remaining terms. Furthermore, the identification (87) is preserved under all symmetry transformations, upon use of the constraints given in equation (83). In particular, the combination  $V_\mu + 2 \tau_\mu S$  does not transform under supersymmetry. It is therefore not needed to realize the supersymmetry algebra off-shell. With the aim of deriving an irreducible multiplet we shall therefore eliminate  $V_\mu$ , using (87). This sets the  $R$ -symmetry parameter  $\rho = \text{const}$  in (80).

Performing the above manipulations, we end up with the following transformation rules for the complete non-relativistic off-shell multiplet

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2} \bar{\epsilon}_+ \gamma^0 \psi_{\mu+}, \\
\delta e_\mu{}^a &= \frac{1}{2} \bar{\epsilon}_+ \gamma^a \psi_{\mu-} + \frac{1}{2} \bar{\epsilon}_- \gamma^a \psi_{\mu+}, \\
\delta m_\mu &= \bar{\epsilon}_- \gamma^0 \psi_{\mu-}, \\
\delta\psi_{\mu+} &= D_\mu \epsilon_+ + \gamma_0 \epsilon_+ S \tau_\mu + \gamma_0 \psi_{\mu+} \rho, \\
\delta\psi_{\mu-} &= D_\mu \epsilon_- - 3 \gamma_0 \epsilon_- S \tau_\mu + \frac{1}{2} \omega_\mu{}^a \gamma_{a0} \epsilon_+ - \gamma_a \epsilon_+ e_\mu{}^a S - \gamma_0 \psi_{\mu-} \rho, \\
\delta S &= -\frac{1}{8} \bar{\epsilon}_+ \gamma^{a0} \hat{\psi}_{a0-}.
\end{aligned} \tag{88}$$

Given that in the non-relativistic case there is only a single (fermionic) equation of motion in the on-shell theory, see equation (63), it is not surprising that the number of auxiliary fields, needed to close the algebra off-shell, is reduced with respect to the relativistic multiplet we started with.

We have explicitly checked that the non-relativistic supersymmetry transformations given above close off-shell, i.e. upon use of the constraints (94)–(96) given below. Note that the commutator algebra closes off-shell in the sense that we do not need the equations of motion (63) to prove closure. To check closure one needs the supersymmetry transformations of the spin- and boost-connection



$$\delta_Q \omega_\mu{}^{ab}(e, \tau, m, \psi_\pm) = \frac{1}{2} \bar{\epsilon}_+ \gamma^{[b} \hat{\psi}^{a]}{}_{\mu-} - S \bar{\epsilon}_+ \gamma^{ab} \psi_{\mu+}, \quad (89)$$

$$\begin{aligned} \delta_Q \omega_\mu{}^a(e, \tau, m, \psi_\pm) &= \frac{1}{4} \bar{\epsilon}_+ \gamma^a \hat{\psi}_{\mu 0-} + \frac{1}{4} e_{\mu b} \bar{\epsilon}_+ \gamma^b \hat{\psi}^a{}_{0-} + S \bar{\epsilon}_+ \gamma^{a0} \psi_{\mu-} \\ &\quad + \bar{\epsilon}_- \gamma^0 \hat{\psi}_\mu{}^a + S \bar{\epsilon}_- \gamma^{a0} \psi_{\mu+}, \end{aligned} \quad (90)$$

as well as the expressions for the gravitino curvatures

$$\begin{aligned} \hat{\psi}_{\mu\nu+} &= 2 \partial_{[\mu} \psi_{\nu]+} - \frac{1}{2} \omega_{[\mu}{}^{ab} \gamma_{ab} \psi_{\nu]+} - 2 \gamma_0 \psi_{[\mu+} \tau_{\nu]} S, \\ \hat{\psi}_{\mu\nu-} &= 2 \partial_{[\mu} \psi_{\nu]-} - \frac{1}{2} \omega_{[\mu}{}^{ab} \gamma_{ab} \psi_{\nu]-} + 6 \gamma_0 \psi_{[\mu-} \tau_{\nu]} S \\ &\quad + \omega_{[\mu}{}^a \gamma_{a0} \psi_{\nu]+} + 2 \gamma_a \psi_{[\mu+} e_{\nu]}{}^a S. \end{aligned} \quad (91)$$

The commutator of two supersymmetry transformations is given by

$$\begin{aligned} [\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_{g.c.t.}(\Xi^\rho) + \delta_J(\Lambda_a{}^b) + \delta_G(\Lambda^a) \\ &\quad + \delta_Z(\Sigma) + \delta_+(\Upsilon_+) + \delta_-(\Upsilon_-), \end{aligned} \quad (92)$$

where the parameters of the transformations on the rhs are given by

$$\begin{aligned} \Xi^\mu &= \frac{1}{2} \bar{\epsilon}_{2+} \gamma^0 \epsilon_{1+} \tau^\mu + \frac{1}{2} (\bar{\epsilon}_{2+} \gamma^a \epsilon_{1-} + \bar{\epsilon}_{2-} \gamma^a \epsilon_{1+}) e^\mu{}_a, \\ \Lambda^{ab} &= -\Xi^\mu \omega_\mu{}^{ab} - S \bar{\epsilon}_{2+} \gamma^{ab} \epsilon_{1+}, \\ \Lambda^a &= -\Xi^\mu \omega_\mu{}^a + S (\bar{\epsilon}_{2+} \gamma^{a0} \epsilon_{1-} + \bar{\epsilon}_{2-} \gamma^{a0} \epsilon_{1+}), \\ \Upsilon_\pm &= -\Xi^\mu \psi_{\mu\pm}, \\ \Sigma &= -\Xi^\mu m_\mu + \bar{\epsilon}_{2-} \gamma^0 \epsilon_{1-}. \end{aligned} \quad (93)$$

We derive the following chain of constraints by supersymmetry variations

$$\xrightarrow{Q_-} \hat{\psi}_{ab-} = 0, \quad (94)$$

$$\hat{R}_{\mu\nu}(H) = 0 \xrightarrow{Q_+} \hat{\psi}_{\mu\nu+} = 0 \xrightarrow{Q_+} \hat{R}_{\mu\nu}{}^{ab}(J) = -4 \varepsilon^{ab} \tau_{[\mu} \hat{D}_{\nu]} S. \quad (95)$$

Note that this is a subset of the constraints given in equations (61)–(63). The Bianchi identities, upon use of the last constraint in equation (95), get the following contributions from the auxiliary field:

$$\hat{R}_{0[a}{}^{b]}(G) = 0, \hat{R}_{ab}{}^c(G) = 2 \varepsilon_{ab} e^{\mu c} \hat{D}_\mu S. \quad (96)$$

With the help of these identities one can show that the supersymmetry variation of the constraint given in equation (94) does not imply any further constraints. The contribution of the auxiliary field in the last equation of (95) ensures that its variation does not lead to additional constraints. The set of constraints given in equations (94)–(96) is thus complete because we varied *all* constraints under supersymmetry. The check here is more complete than in the on-shell case where we did not vary the bosonic equation of motion anymore. Since the on-shell case can be derived from the off-shell formulation, see below, we have also proven consistency of the on-shell formulation.

As a consistency check we note that the above result for the off-shell multiplet entails the two on-shell formulations that were presented earlier in the literature. First of all, by imposing

$$S = 0, \quad (97)$$

we arrive at the result of section 2.3. Secondly, choosing

$$S = \frac{1}{2R}, \quad (98)$$

with  $R$  constant, and related to the cosmological constant by  $\Lambda_{\text{CC}} = -1/R^2$ , we reproduce the on-shell Newton–Hooke supergravity theory of [30]. The  $1/R^2$  corrections w.r.t. the flat case are hidden in the curvatures, e.g. the bosonic equation of motion for Newton–Hooke supergravity is still given by equation (63), but  $\hat{R}_{\mu\nu}{}^a(G)$  now contains additional terms of order  $1/R$ .

This concludes the discussion of the off-shell formulation of non-relativistic three-dimensional Newton–Cartan supergravity.

#### 4. Conclusions and outlook

In this paper, we have discussed how generally covariant, non-relativistic (super)gravity theories can be obtained from relativistic ones via a procedure that implements the non-relativistic limit. The method extends the Inönü–Wigner contraction, that yields a non-relativistic spacetime symmetry algebra starting from a relativistic one, to an irreducible (super) multiplet of fields representing the algebra. In applying this method, special care has to be taken of various consistency checks to avoid divergences in geometric quantities and transformation rules. We have shown how this procedure can be used to obtain torsionless Newton–Cartan gravity from general relativity, three-dimensional on-shell Newton–Cartan supergravity from relativistic on-shell supergravity and how it can be used to obtain non-relativistic superparticle actions from relativistic ones.

We would also like to remind the reader that if we are not interested in *super*gravity we are free to impose the weaker ‘kinematical’ constraint (20), which might lead to twistless torsionfull Newton–Cartan structures. However, also in this case we do get restrictions on the gauge-field  $\tau_\mu$ , see equations (32) and (33). In contrast, the limit discussed in [29] does not lead to any restriction of  $\tau_\mu$ .

In contrast to methods that are based on the gauging of algebras, the limiting procedure has the advantage that it can be extended to the case in which the relativistic (super)multiplet contains (auxiliary) fields that are not associated to gauge-fields of the underlying spacetime symmetry algebra. As an example, we have derived a new off-shell formulation of three-dimensional Newton–Cartan supergravity containing a real auxiliary scalar  $S$ .

Several extensions of this work can be considered. For example, now that the general limiting procedure has been defined, it would be interesting to apply it to a specific version of off-shell 4D  $\mathcal{N} = 2$  Poincaré supergravity to obtain 4D off-shell Newton–Cartan supergravity. We note that the examples considered in this paper only dealt with pure Newton–Cartan (super)gravity theories. It would be interesting to extend the procedure to matter coupled (super)gravity theories (see [18, 29] for similar ideas applied to condensed matter theories). Another interesting extension is to consider relativistic theories whose underlying symmetry algebra is different from the Poincaré (super)algebra, such as conformal ones. In particular, one can define a contraction from the relativistic (super)conformal algebra to the Galilean (super)conformal algebra, that has been discussed in the context of non-relativistic limits of AdS/CFT [45] and flat space holography (see e.g. [46]). One could try to extend this contraction to a vielbein (super)multiplet of the (super)conformal algebra and in this way find background theories for (supersymmetric versions of) the Galilean conformal algebra, see e.g.

[47–49]. Note that it is not obvious how to do this via a gauging procedure, as the Galilean conformal algebra does not allow for the type of central extension that was crucial in the gauging of the Bargmann algebra.

As we mentioned in the introduction this work only considers ‘particle’ backgrounds. It would be interesting to extend the limiting procedure to the case of non-relativistic branes. Here, one would first have to find a suitable extension of the Poincaré algebra whose contraction leads to extended stringy Galilei algebras [50].

Another interesting extension is to consider other limits than the non-relativistic one, such as the ultra-relativistic limit<sup>7</sup>. At the level of algebras, the latter yields a contraction of the Poincaré algebra to the Carroll algebra. It would be interesting to see whether this algebra contraction can be extended to a relativistic vielbein multiplet and to check whether such a limit can e.g. be used to derive the recently constructed action for a Carroll (super-)particle in a curved background [51, 52].

We should stress that, as presented in this work, it is not clear whether every algebra contraction can be translated into a contraction at the level of the field theory representing that algebra. Moreover, certain non-relativistic symmetry algebras cannot be viewed as contractions of relativistic ones. An example of such an algebra is given by the Schrödinger algebra. On the other hand, the Bargmann and the Schrödinger algebra can be obtained as light-like reductions of relativistic algebras [53, 54]. Perhaps one can define a different sort of contraction or limiting procedure related to such kind of reductions which would give rise to (torsional) Newton–Cartan structures as presented in [23, 25]. In view of the recent applications of torsional Newton–Cartan geometry in non-relativistic holography [9–13], it would be interesting to investigate this case in more detail.

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